<u>Recall</u>: • $\psi(x) = \sum_{n \le x} \Lambda(n)$

• PNT $\Longleftrightarrow \psi(x) \sim x$.

• For $Re(s) > 1$, we have $-\frac{\zeta'}{\zeta}(s) = \sum_{n \ge 1} \frac{\Lambda(n)}{n^s}$

Truncated Perron: $c = 1 + \frac{1}{\log x}$, $2 \le T \le 2x$

$$\psi(x) = \frac{1}{2\pi i} \int_{c-iT}^{c+iT} \left(-\frac{\zeta'}{\zeta}(s)\right) \cdot x^s \frac{ds}{s} + O\left(\frac{x (\log x)^2}{T}\right).$$

Next we prove an explicit formula for $\psi(x)$:

<u>Theorem</u> (Explicit formula for $\psi(x)$)

Let $2 \le T \le 2x$. Then

$$\psi(x) = x - \sum_{|\text{Im}\,\rho| \le T} \frac{x^\rho}{\rho} + O\left(\frac{x (\log x)^2}{T}\right).$$

Note: This formula shows distribution of primes closely related to location of zeros of $\zeta(s)$.

<u>Pf</u>: Recall that from truncated Perron formula, we have that for $c = 1 + \frac{1}{\log x}$, $2 \le T \le 2x$

$$\psi(x) = \frac{1}{2\pi i}\int_{c-iT}^{c+iT}\left(-\frac{\zeta'}{\zeta}(s)\right)x^s\,\frac{ds}{s} + O\left(\frac{x(\log x)^2}{T}\right)$$

We want to estimate the integral $\frac{1}{2\pi i}\int_{c-iT}^{c+iT} x^s \frac{\zeta'}{\zeta}(s)\frac{ds}{s}$ by moving the line of integration and applying residue theorem. Therefore we need to find the poles of $\frac{\zeta'}{\zeta}(s)\frac{x^s}{s}$ and compute the residues.

x^s has no poles in the complex plane

$\frac{1}{s}$ has a simple pole at 0, no other poles.

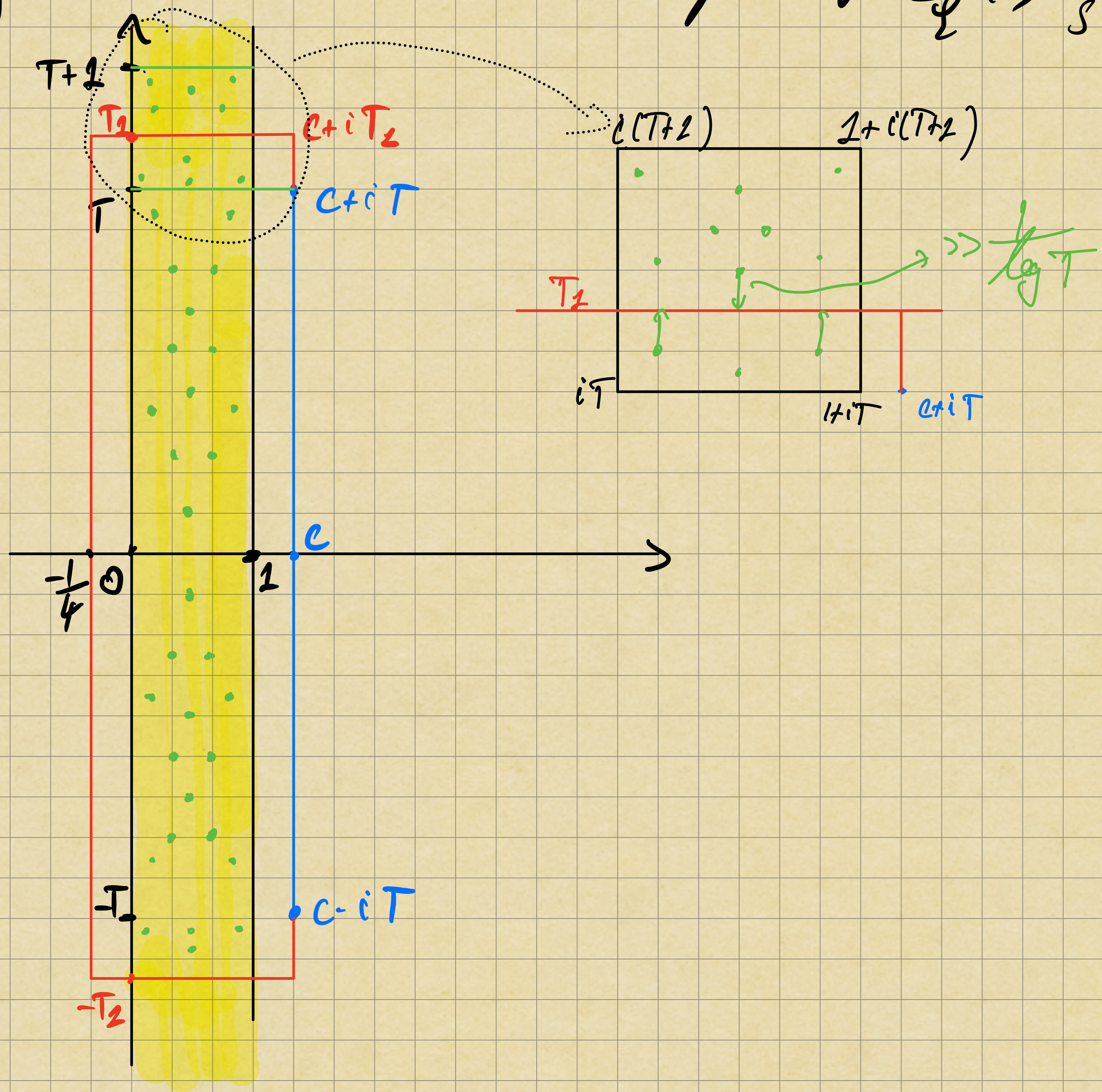
$\frac{\zeta'}{\zeta}(s)$ has a simple pole at 1 and a simple pole at each zero of $\zeta(s)$.

Residue at $s=1$ is $\lim_{s\to 1}(s-1)\frac{\zeta'}{\zeta}(s)\frac{x^s}{s} = -x$

Residue at $s=0$ is $\lim_{s\to 0} x^s \frac{\zeta'}{\zeta}(s) = \frac{\zeta'}{\zeta}(0)$ (a constant)

If $\zeta(s)$ has zero ρ with multiplicity m_ρ, then $\lim_{s\to\rho}(s-\rho)\frac{\zeta'}{\zeta}(s)\frac{x^s}{s} = m_\rho\frac{x^\rho}{\rho}$.

We want to carefully choose our box of integration such that we avoid poles of $\frac{\xi'}{\xi}(s)\frac{X^s}{s}$.



From last time, we know there are $O(\log T)$ zeros of $\xi(s)$ with $\operatorname{Im}(\rho) \in [T, T+1]$.
Therefore there exists $T_1 \in [T, T+1]$ such that all zeros ρ of $\xi(s)$ satisfy $|\operatorname{Im}\rho - T_1| \gg \frac{1}{\log T}$

Can put absolute values

because if ρ is a zero, so is $\bar{\rho}$

We integrate $\frac{x^s}{s}\frac{\zeta'}{\zeta}(s)$ along the box with corners $c-iT_2$, $c+iT_2$, $-\frac{1}{4}+iT_2$ and $-\frac{1}{4}-iT_2$

This gives
$$\frac{1}{2\pi i}\left(\int_{c-iT_2}^{c+iT_2} x^s\frac{\zeta'}{\zeta}(s)\frac{ds}{s}+\int_{c-iT_2}^{-\frac{1}{4}+iT_2} x^s\frac{\zeta'}{\zeta}(s)\frac{ds}{s}+\int_{-\frac{1}{4}+iT_2}^{-\frac{1}{4}-iT_2} x^s\frac{\zeta'}{\zeta}(s)\frac{ds}{s}+\int_{-\frac{1}{4}+iT_2}^{-c+iT_2} x^s\frac{\zeta'}{\zeta}(s)\frac{ds}{s}\right)=-x+\frac{\zeta'}{\zeta}(0)+\sum_{\rho:|\mathrm{Im}\,\rho|\le T_2}\frac{x^\rho}{\rho}$$

residue at 1 res at 0 residue at ρ.

From last time: $\frac{\zeta'}{\zeta}(\sigma+iT_2)=O((\log T)^2)$, for $-\frac{1}{4}\le\sigma\le c$.

Therefore
$$\int_{c+iT_2}^{-\frac{1}{4}+iT_2}\frac{x^s}{s}\frac{\zeta'}{\zeta}(s)\,ds \ll \frac{x^c(\log T)^2}{T}\ll\frac{x(\log x)^2}{T}$$
(and similarly for $\int_{-\frac{1}{4}-iT_2}^{c+iT_2}$).

Now,
$$\int_{-\frac{1}{4}+iT_2}^{-\frac{1}{4}-iT_2} x^s\frac{\zeta'}{\zeta}(s)\frac{ds}{s}\ll\frac{(\log T)^2}{x^{1/4}}\int_{-T_2}^{T_2}\frac{dt}{1+|t|}\ll\frac{(\log T)^3}{x^{1/4}}$$

we used $\frac{\zeta'}{\zeta}(-\frac{1}{4}+it)\ll\log(1+|t|)^2$ from last time.

This implies $\displaystyle\frac{1}{2\pi i}\int_{c-iT_1}^{c+iT_1} x^s \frac{\zeta'}{\zeta}(s)\frac{ds}{s} = -x + \sum_{\rho : |\mathrm{Im}\,\rho| \le T_1} \frac{x^\rho}{\rho}$

$$+ O\left(\frac{(\log T)^3}{x^{1/4}} + \frac{x}{T}(\log x)^2\right).$$

Finally, there are $O(\log T)$ zeros ρ with $T \le |\mathrm{Im}\,\rho| \le T_1$, and each contributes $O\left(\frac{x}{T}\right)$.

Hence $\displaystyle\sum_{\rho : T \le |\mathrm{Im}\,\rho| \le T_1} \frac{x^\rho}{\rho} \ll \frac{x \log T}{T}$.

Also, $\displaystyle\int_{c+iT}^{c+iT_1} \frac{x^s}{s}\frac{\zeta'}{\zeta}(s)\,ds \ll \frac{x}{T}\log x$.

(we used $\displaystyle\left|\frac{\zeta'}{\zeta}(c+it)\right| \le -\frac{\zeta'}{\zeta}(c) = \sum_{n\ge 2}\frac{\Lambda(n)}{n^c}$

$$\ll \sum_{n \ge 2} \frac{1}{n^{\frac{1+c}{2}}} = \zeta\left(\frac{1+c}{2}\right) \ll \log x,$$

recall $c = 1 + \frac{1}{\log x}$).

Hence we have indeed

$$\psi(x) = -\frac{1}{2\pi i}\int_{c-iT_1}^{c+iT_1} \frac{x^s}{s}\frac{\zeta'}{\zeta}(s)\,ds + O\left(\frac{x(\log x)^2}{T}\right)$$

$$= x - \sum_{|\mathrm{Im}\,\rho| \le T_1} \frac{x^\rho}{\rho} + O\left(\frac{x(\log x)^2}{T} + \frac{(\log T)^3}{x^{1/4}}\right)$$

$$= x - \sum_{|\operatorname{Im}\rho| \le T} \frac{x^\rho}{\rho} + O\left(\frac{x(\log x)^2}{T}\right) \quad \square$$

The explicit formula shows that the distribution of primes is closely related to the location of zeros of $\zeta(s)$. Next we see what we obtain under the most optimistic assumption about the location of zeros (RH).

<u>Theorem</u> (Error term under RH)
The Riemann hypothesis is true if and only if

$$\psi(x) = x + O(x^{\frac{1}{2}+\varepsilon}), \forall \varepsilon > 0.$$

<u>Proof</u>: "$\Rightarrow$" Applying Explicit Formula with $x = T$, we have

$$\psi(x) = x - \sum_{\rho : |\operatorname{Im}\rho| \le x} \frac{x^\rho}{\rho} + O((\log x)^2).$$

Now suppose all non-trivial zeros have $\operatorname{Re}(\rho) = \frac{1}{2}$.
For $n \ge 5$, there are $O(\log n)$ zeros with $|\operatorname{Im}\rho| \in [n, n+1]$, and each one contributes $O\left(\frac{x^{1/2}}{n}\right)$.

Therefore $\psi(x) = x + O\left(1 + x^{1/2} \sum_{5 \le n \le x} \frac{\log n}{n} + (\log x)^2\right)$.

$$= x + O(x^{1/2} (\log x)^2).$$

✓

"$\Longleftarrow$" By partial summation (initially in $\mathrm{Re}(s) > 1$)

$$-\frac{\zeta'}{\zeta}(s) = \sum \Lambda(n) n^{-s} = s \int_1^{\infty} \psi(y) y^{-s-1} dy .$$

Suppose $\psi(x) = x + R(x)$, where $R(x) \ll x^{\frac{1}{2}+\varepsilon}$.

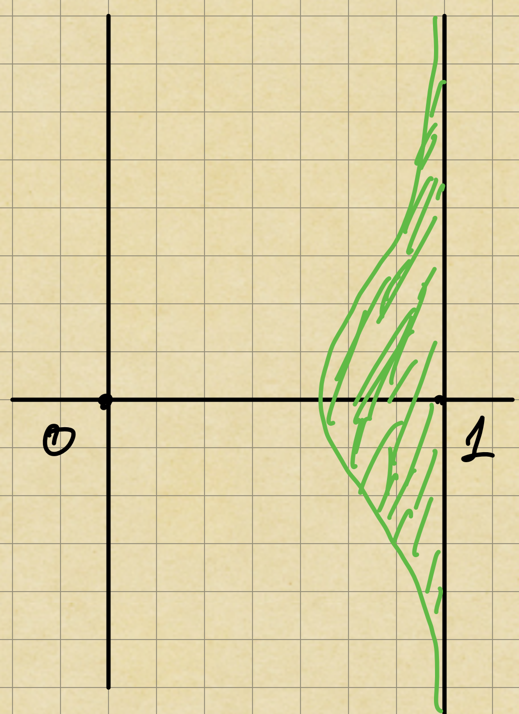
$$\Rightarrow -\frac{\zeta'}{\zeta}(s) = \frac{s}{s-1} + s \int_1^{\infty} R(x) x^{-s-1} dx .$$

Then integral is holomorphic for $\mathrm{Re}(s) > \frac{1}{2}+\varepsilon$, so $\zeta(s)$ is zero-free in this region. $\square$

Unfortunately RH is completely out of reach, but we have some understanding for zero-free regions.

Theorem (Zero-free region for $\zeta(s)$)

There exists a constant $c>0$ such that if $\rho=\sigma+it$ is a non-trivial zero of $\zeta(s)$, then $\sigma \le 1-\frac{c}{\log(2+|t|)}$.



Proof: We have the following key elementary identity:

For $\alpha \in \mathbb{R}$, $3+4\cos\alpha+\cos 2\alpha = 2(1+\cos\alpha)^2 \ge 0$.

Note that for $\sigma>1$,

$$\operatorname{Re}\left(-\frac{\zeta'}{\zeta}(\sigma+it)\right)=\operatorname{Re}\left(\sum_{n\ge 2}\frac{\Lambda(n)}{n^{\sigma+it}}\right)$$

$$=\sum_{n\ge 1}\frac{\Lambda(n)}{n^{\sigma}}\cos(t\log n).$$

Therefore, for $\sigma > 1$, we have

$$\mathrm{Re}\left(-3\frac{\zeta'}{\zeta}(\sigma) - 4\frac{\zeta'}{\zeta}(\sigma+it) - \frac{\zeta'}{\zeta}(\sigma+2it)\right)$$

$$= \sum_n \frac{\Lambda(n)}{n^\sigma}\Big(\underbrace{3+4\cos(t\log n)+\cos(2t\log n)}_{\geq 0}\Big)$$

$$\geq 0.$$

$C_0 > 0$ some constant we choose later

Let $1<\sigma<2$ and $t \geq C_0$. Then by partial fraction expansion of $\frac{\zeta'}{\zeta}(s)$, we have for $s=\sigma+it$

$$-\frac{\zeta'}{\zeta}(\sigma+it) = -\sum_{|s-\rho|<1} \frac{1}{s-\rho} + O(\log t).$$

Note that $\mathrm{Re}\left(\frac{1}{s-\rho}\right) = \frac{\mathrm{Re}(s-\rho)}{|s-\rho|^2} > 0.$

Also, recall that $-\frac{\zeta'}{\zeta}(\sigma) = \frac{1}{\sigma-1} + O(1).$

Now let a zero $\rho_0 = \beta+it$, with $t \geq C_0$.

Then $0 \leq \mathrm{Re}\left(-3\frac{\zeta'}{\zeta}(\sigma) - 4\frac{\zeta'}{\zeta}(\sigma+it) - \frac{\zeta'}{\zeta}(\sigma+2it)\right)$

$$= \frac{3}{\sigma-1} + O(1) - \mathrm{Re}\left(\frac{4}{\sigma+it-\rho_0}\right) - \underbrace{\sum_{\substack{\rho:|\rho-s|<1\\ \rho\neq\rho_0}} \mathrm{Re}\left(\frac{1}{s-\rho}\right)}_{\geq 0}$$

$$- \underbrace{\sum_{\rho:|\rho-(\sigma+2it)|<1} \mathrm{Re}\left(\frac{1}{\sigma+2it-\rho}\right)}_{\geq 0} + O(\log t).$$

Hence there exists an absolute constant C_1 such that

$$\frac{3}{\sigma-1} - \frac{4}{\sigma-\beta} + C_1 \log t \geq 0.$$

Choose $\sigma = 1 + \frac{\delta}{\log t}$, for some $\delta > 0$.

Then $\beta < 1 + \frac{\delta}{\log t} - \frac{4\delta}{(3+C_1\delta)\log t}$

Let $\delta = \frac{1}{3C_1}$, we have $\beta < 1 - \frac{1}{15 C_1 \log t}$.

This proves theorem if $\operatorname{Im} \rho \geq C_0$. By symmetry, it follows for all ρ with $\operatorname{Im} \rho \leq -C_0$.

We have finitely many zeros with $|\operatorname{Im} \rho| \leq C_0$, and all away bounded away from $\{\operatorname{Re}(s) = 1, \operatorname{Im}(s) \leq C_0\}$. So theorem follows if $c > 0$ sufficiently small. $\square$

(We choose C_0 such that $\zeta(1+it) \neq 0$, for $0 < |t| \leq C_0$.

Such C_0 exists because $\zeta(s)$ has pole at 1.

$$\zeta(s) = \frac{1}{s-1} + O(1)$$